Moments of Gaußian β Ensembles via Map Enumeration

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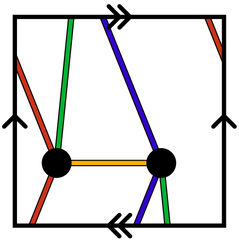
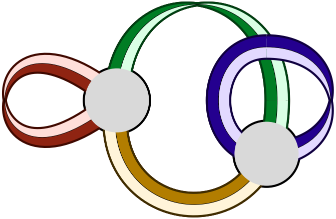
April 10, 2012





Outline

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The Problem

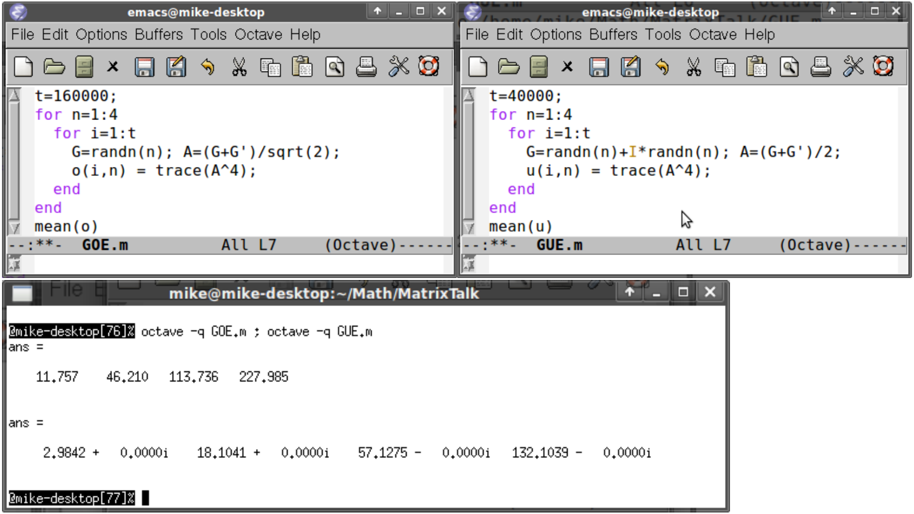
Q: If A is selected from an $n \times n$ Gaußian β ensemble, how can we interpret symmetric functions of the eigenvalues of A?

$$\mathrm{E}(\mathrm{tr}(A^4)) \qquad \mathrm{E}(J_{[4,2]}^{(2/\beta)}(A)) \qquad \text{etc.}$$

A: With suitable normalization $\mathrm{E}(p_\theta(A))$ is a polynomial with non-negative integer coefficients in n and $b = \frac{2}{\beta} - 1$ for every partition θ.

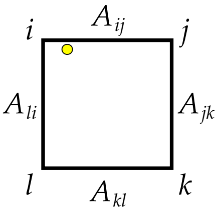
The coefficients can be obtained by counting appropriately weighted maps.

$E(\mathrm{tr}(A^4))$ for A selected from a Gaußian Ensemble



What's actually being computed?

$$\mathrm{tr}(A^4) = \sum_{i,j,k,l} A_{ij} A_{jk} A_{kl} A_{li}$$



$$\begin{aligned}\mathrm{E}\left(\mathrm{tr}(A^4)\right) = 4!\binom{n}{4}\mathrm{E}(A_{12}A_{23}A_{34}A_{41}) + 3!\binom{n}{3}\mathrm{E}(4A_{11}A_{12}A_{23}A_{31}) \\ + 3!\binom{n}{3}\mathrm{E}(2A_{12}A_{21}A_{13}A_{31}) + 2!\binom{n}{2}\mathrm{E}(2A_{11}A_{12}A_{22}A_{21}) \\ + 2!\binom{n}{2}\mathrm{E}(A_{12}A_{21}A_{12}A_{21}) + 2!\binom{n}{2}\mathrm{E}(4A_{11}A_{11}A_{12}A_{21}) \\ + 1!\binom{n}{1}\mathrm{E}(A_{11}A_{11}A_{11}A_{11})\end{aligned}$$

Wick's Lemma

If A an element of a GUE, then

$$E(A_{ii}^{2k}) = (2k-1)!! \qquad E(A_{ij}^{k} A_{ji}^{l}) = k!\delta_{kl}$$

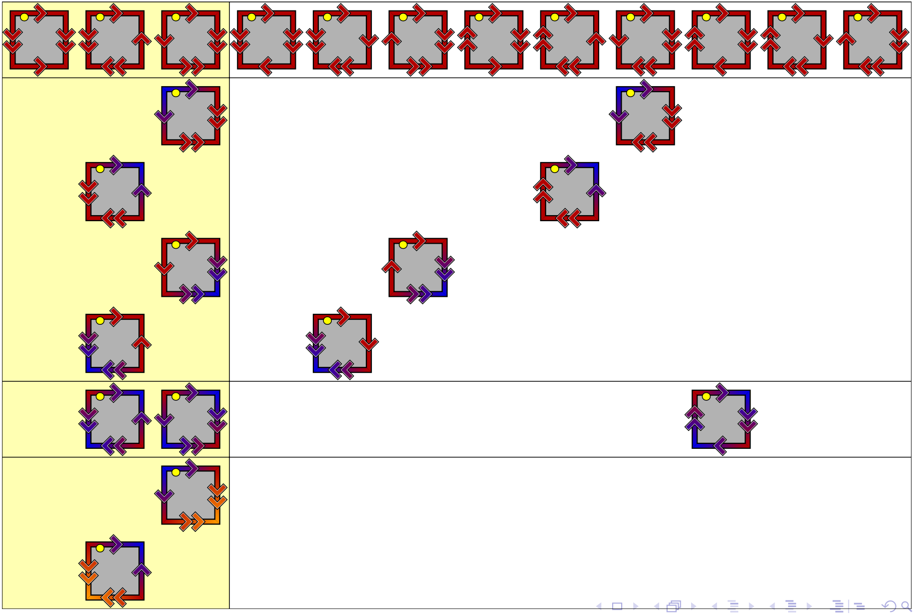
If A an element of a GOE, then

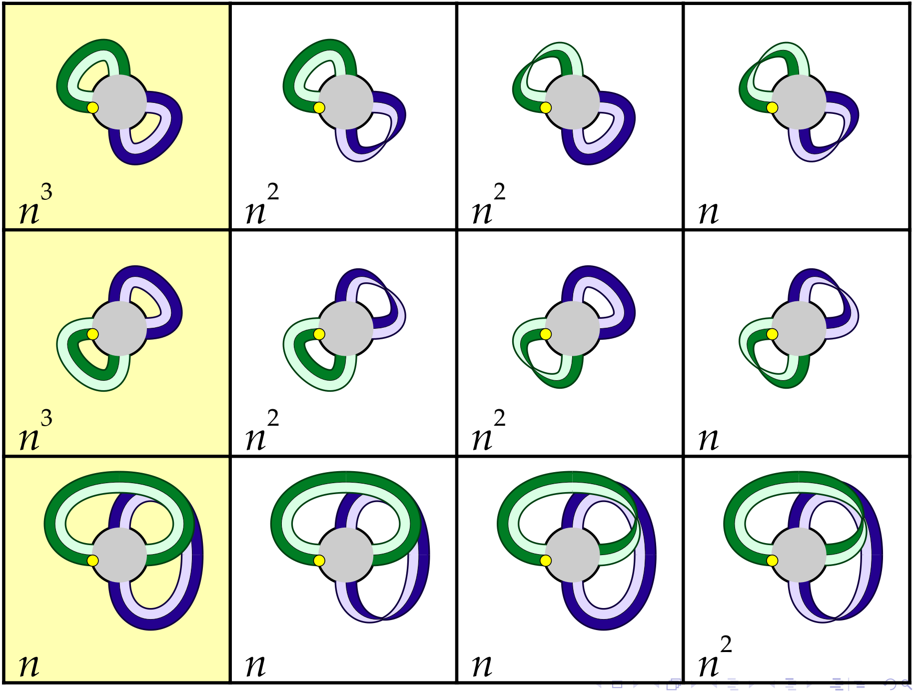
$$E(A_{ii}^{2k}) = 2^k(2k-1)!! \qquad E(A_{i,j}^{k} A_{j,i}^{l}) = (k+l-1)!! \text{ if } k+l \text{ is even}$$

$$\sum_{p \text{ a painting}} \#\{\text{pairings consistent with } p\}$$

$$= \sum_{m \text{ a pairing}} \#\{\text{paintings consistent with } m\}$$

Count the polygon glueings in 2 different ways




n^3
n^2
n^2
n
n^3
n^2
n^2
n
n
n
n
n^2

Outline

Graphs, Surfaces, and Maps

Definition

A **surface** is a compact 2-manifold without boundary.

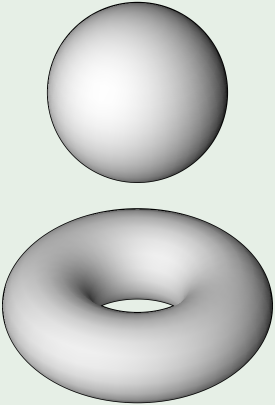
Definition

A **graph** is a finite set of *vertices* together with a finite set of *edges*, such that each edge is associated with either one or two vertices. (It may have loops / parallel edges.)

Definition

A **map** is a 2-cell embedding of a graph in a surface.

Example



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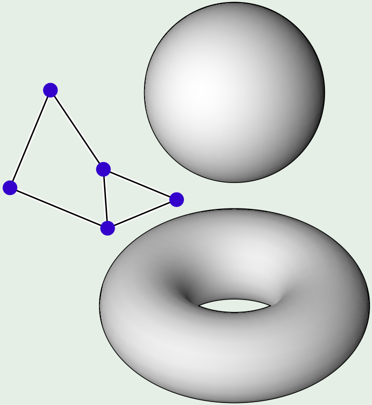

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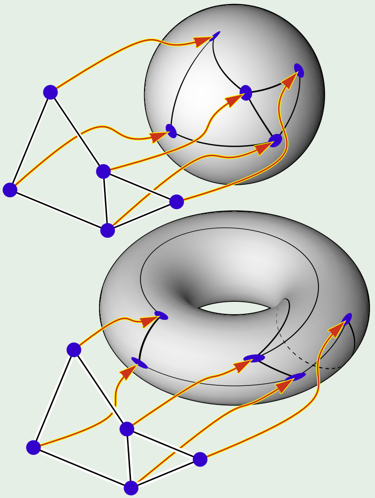
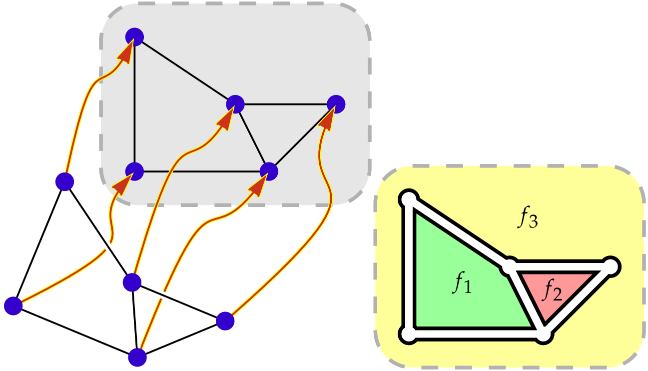


Faces

Once a graph is drawn, the unused portion of the paper is split into faces.



A **map** is a graph together with an embedding in a surfaces. It is defined by its vertices, edges, and faces.

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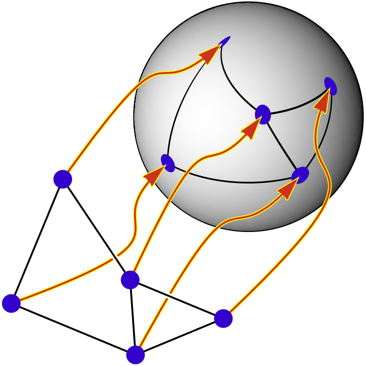
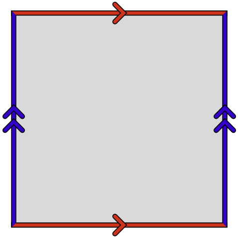


For symmetry, the outer face is thought of as part of a sphere.

A **map** is a graph together with an embedding in a surfaces. It is defined by its vertices, edges, and faces.

Polygon Glueings = Graphs in Surfaces = Maps

Identifying the edges of a polygon creates a surface, with the boundary of the polygon becoming a graph embedded in that surface.



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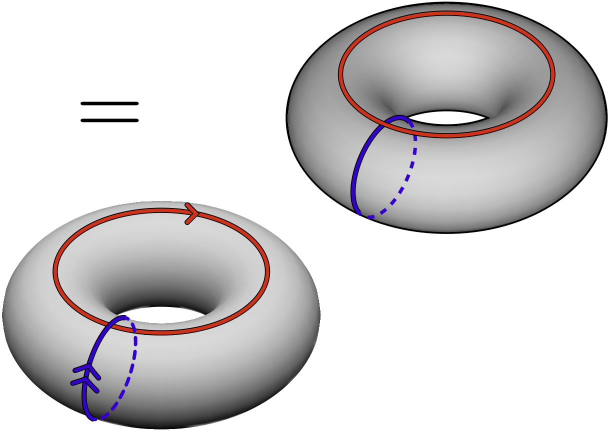

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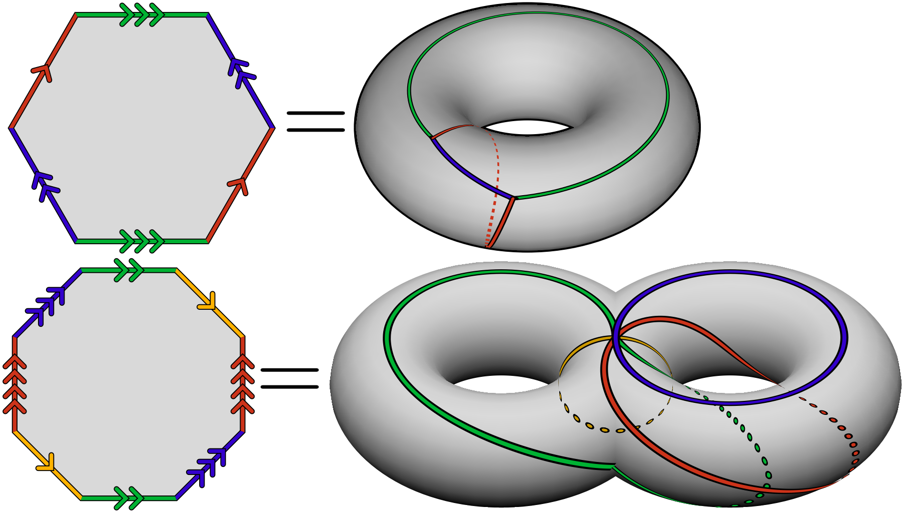
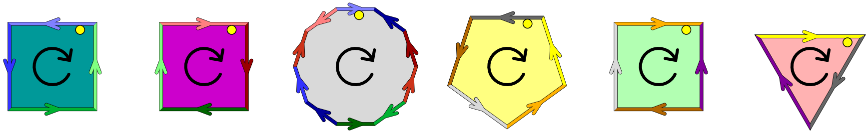
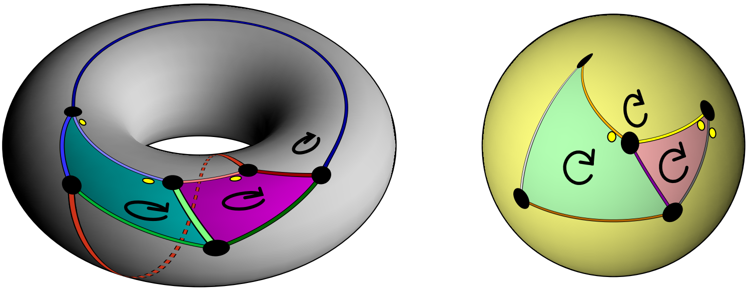


Polygon Glueings = Graphs in Surfaces = Maps

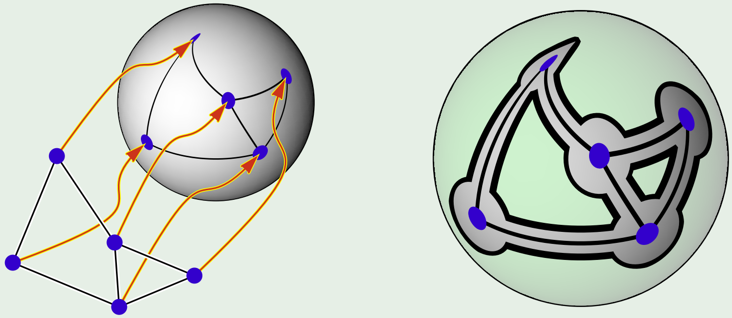
Identifying edges of multiple polygons constructs a map (or maps) with multiple faces.





Ribbon Graphs

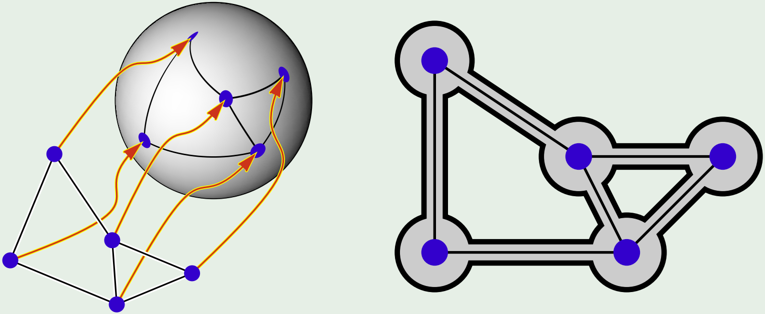
Example



The homeomorphism class of an embedding is determined by a neighbourhood of the graph.

Ribbon Graphs

Example



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Ribbon Graphs

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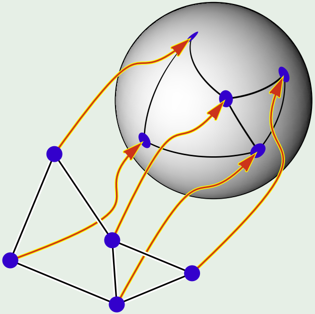
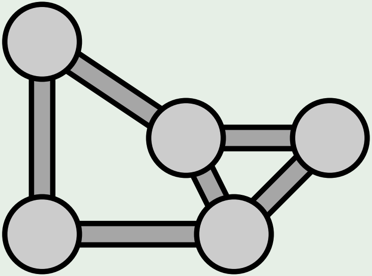
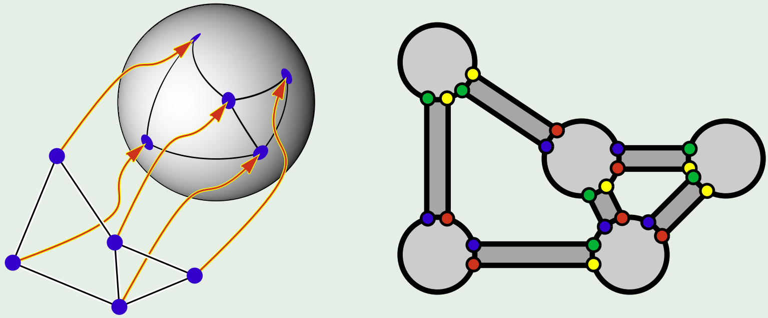




Neighbourhoods of vertices and edges can be replaced by discs and ribbons to form a ribbon graph.

Flags

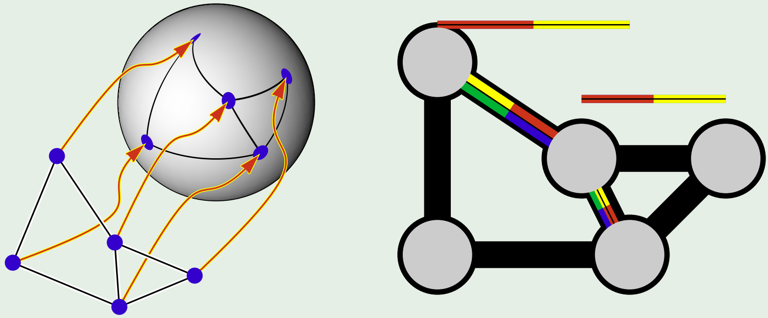
Example



The boundaries of ribbons determine flags.

Flags

Example



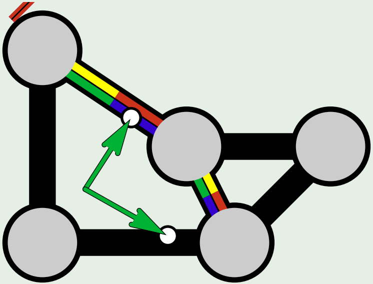
The boundaries of ribbons determine flags, and these can be associated with quarter edges.

Rooted Maps

Definition

A **rooted map** is a map together with a distinguished orbit of flags under the action of its automorphism group.

Example



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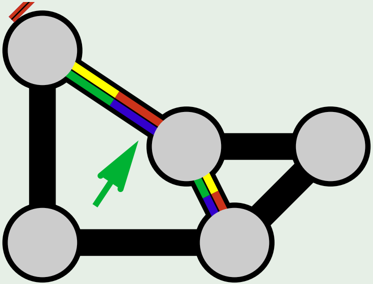

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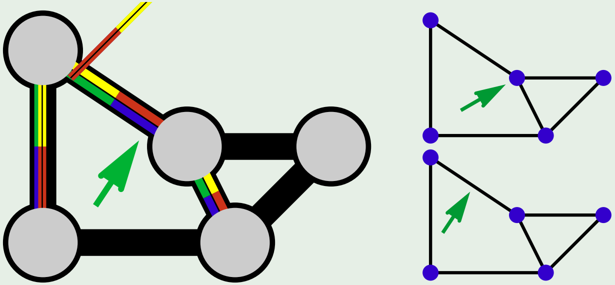


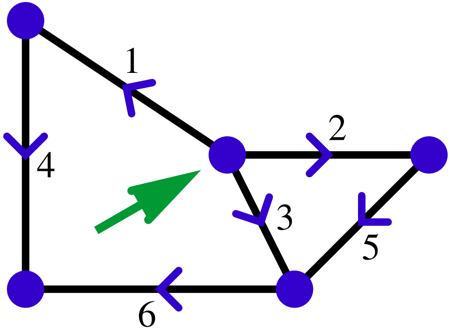
Outline

How to enumerate maps with symmetric functions

- Instead of counting rooted maps, we can count labelled hypermaps. This adds easily computable multiplicities.
- Labelled counting problems are turned into problems involving counting factorizations.
- These can be answered via character theory. (for $\beta \in \{1, 2\}$)
- Appropriate characters appear as coefficients of symmetric functions.
- Standard enumerative techniques restrict the solution to connected maps and remove factors introduced by the labelling.

Encoding Orientable Maps

1. Orient and label the edges.
2. This induces labels on flags.
3. Clockwise circulations at each vertex determine ν.
4. Face circulations are the cycles of $\epsilon\nu$.



$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')(6\ 6')$$
$$\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6')$$
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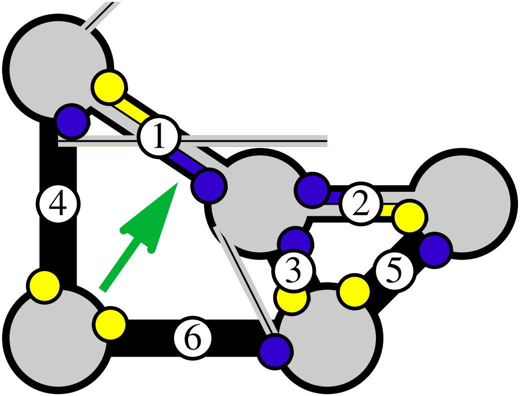

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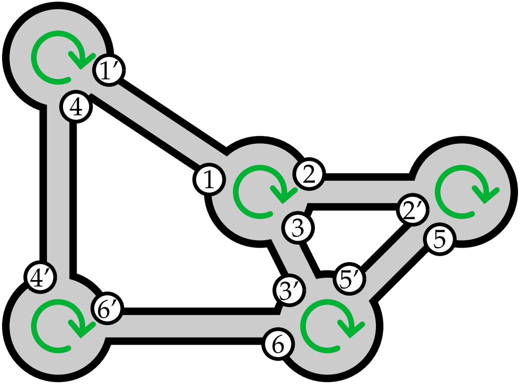

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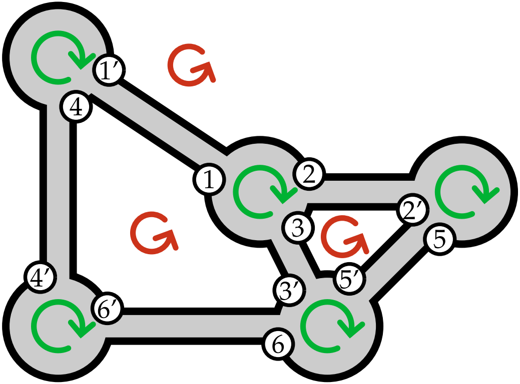

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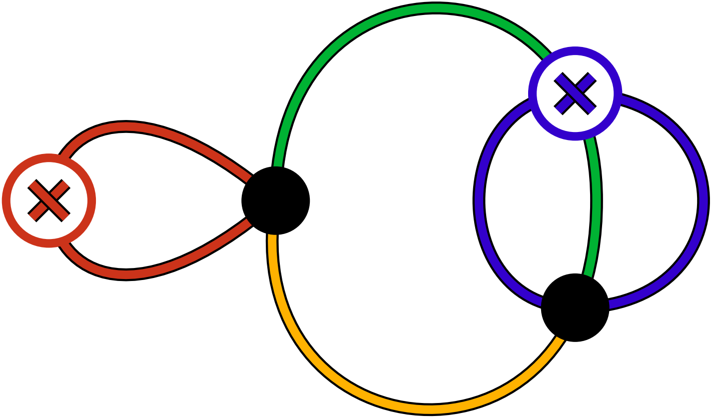
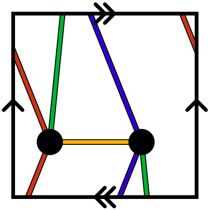
Encoding all Maps

Equivalence classes can be encoded by perfect matchings of flags.





Start with a ribbon graph.

Encoding all Maps

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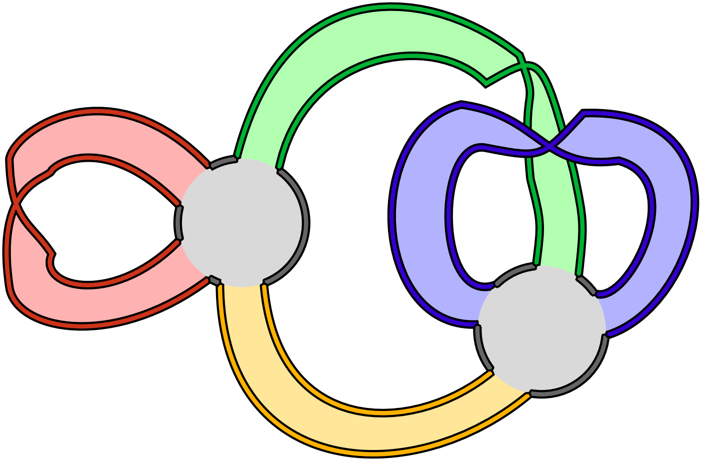

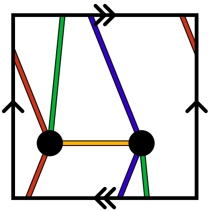

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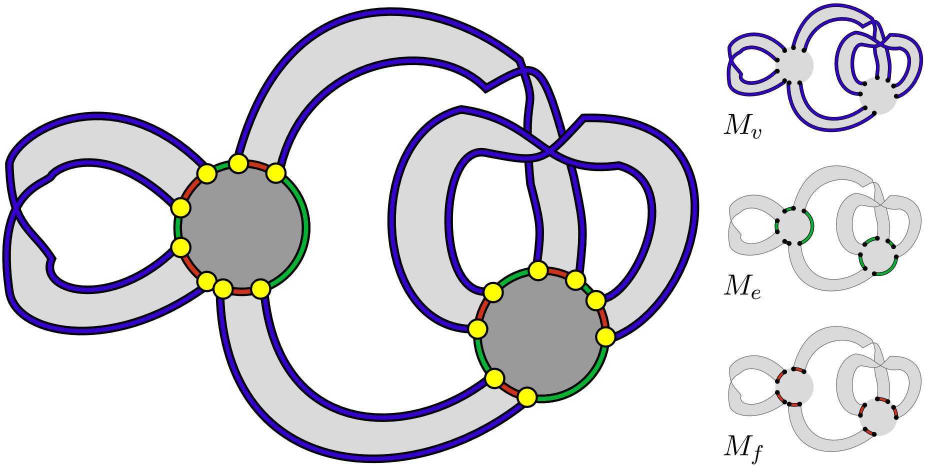

Ribbon boundaries determine 3 perfect matchings of flags.

Encoding all Maps

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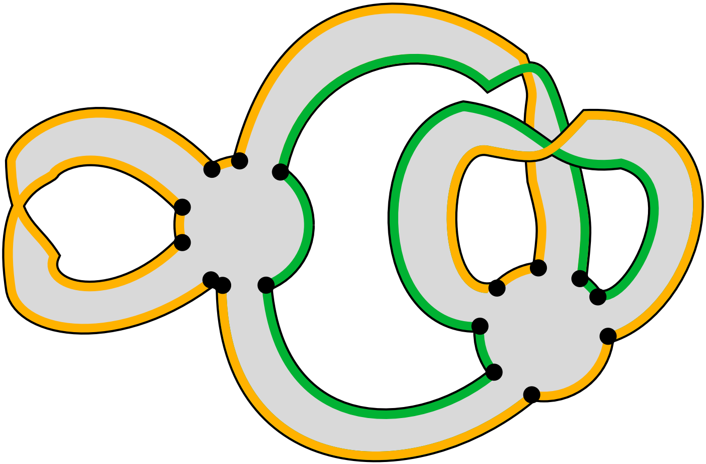

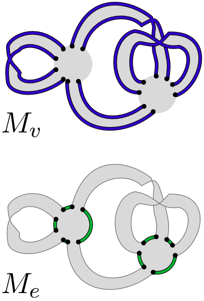

Pairs of matchings determine, faces,

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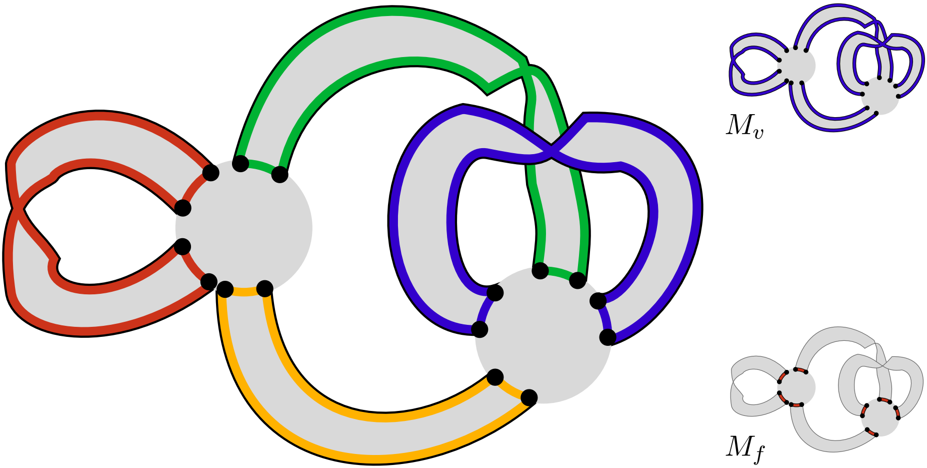

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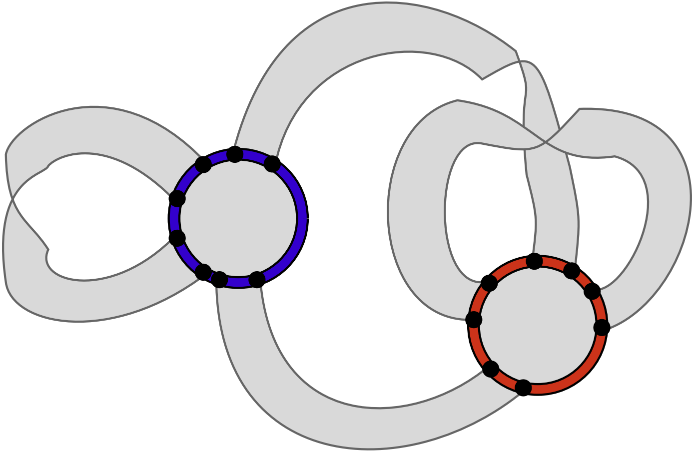
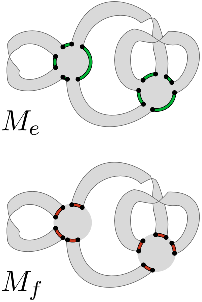
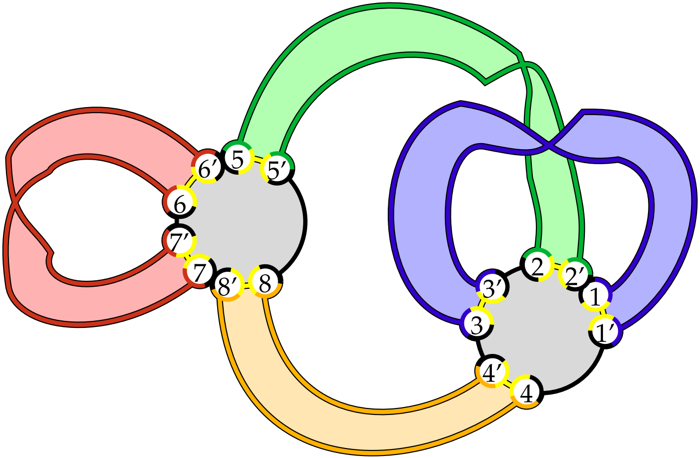
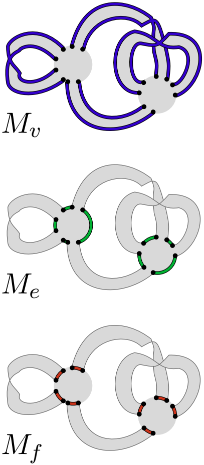




Pairs of matchings determine, faces, edges, and vertices.

Encoding all Maps





$$M_v = (1\,3)(1'\,3')(2\,5)(2'\,5')(4\,8')(4'\,8)(6\,7)(6'\,7')$$

$$M_e = (1\,2')(1'\,4)(2\,3')(3\,4')(5\,6')(5'\,8)(6\,7')(7\,8')$$

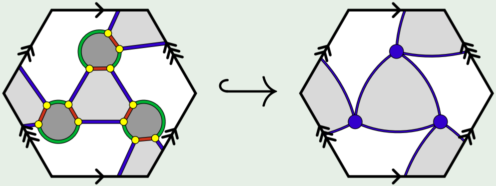
$$M_f = (1\,1')(2\,2')(3\,3')(4\,4')(5\,5')(6\,6')(7\,7')(8\,8')$$

Hypermaps

Generalizing the combinatorial encoding, an arbitrary triple of perfect matchings determines a **hypermap** when the triple induces a connected graph, with cycles of $M_e \cup M_f$, $M_e \cup M_v$, and $M_v \cup M_f$ determining vertices, hyperfaces, and hyperedges. ▸ Example

Hypermaps both **specialize** and generalize maps.

Example



Hypermaps can be represented as face-bipartite maps.

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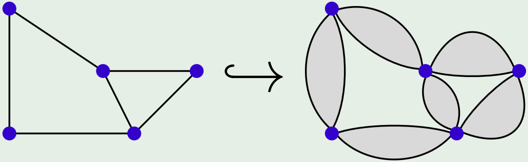
Hypermaps both specialize and **generalize** maps.

Example



Maps can be represented as hypermaps with $\epsilon = [2^n]$.

The Hypermap Series

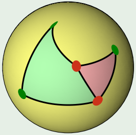
Definition

The **hypermap series** for a set $\mathcal{H}$ of hypermaps is the combinatorial sum

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \sum_{\mathfrak{h} \in \mathcal{H}} \mathbf{x}^{\nu(\mathfrak{h})} \mathbf{y}^{\phi(\mathfrak{h})} \mathbf{z}^{\epsilon(\mathfrak{h})}$$

where $\nu(\mathfrak{h})$, $\phi(\mathfrak{h})$, and $\epsilon(\mathfrak{h})$ are the vertex-, hyperface-, and hyperedge-degree partitions of $\mathfrak{h}$. ▸ Example

Example

Rootings of  contribute $12\,(x_2^3\, x_3^2)\,(y_3\, y_4\, y_5)\, z_2^6$ to the sum.

Explicit Formulae

The hypermap series can be computed explicitly when $\mathcal{H}$ consists of orientable hypermaps or all hypermaps.

Theorem (Jackson and Visentin - 1990)

When $\mathcal{H}$ is the set of orientable hypermaps,

$$H_{\mathcal{O}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 0\Big) = t\frac{\partial}{\partial t} \ln \left(\sum_{\theta \in \mathscr{P}} t^{|\theta|} H_\theta s_\theta(\mathbf{x}) s_\theta(\mathbf{y}) s_\theta(\mathbf{z}) \right)\Bigg|_{t=1}.$$

Theorem (Goulden and Jackson - 1996)

When $\mathcal{H}$ is the set of all hypermaps (orientable and non-orientable),

$$H_{\mathcal{A}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 1\Big) = 2t\frac{\partial}{\partial t} \ln \left(\sum_{\theta \in \mathscr{P}} t^{|\theta|} \frac{1}{H_{2\theta}} Z_\theta(\mathbf{x}) Z_\theta(\mathbf{y}) Z_\theta(\mathbf{z}) \right)\Bigg|_{t=1}.$$

Explicit Formulae

Comparing these expressions to expectations over Gaußian ensembles gives:

Corollary

When A is taken from an $n \times n$ GUE,

$$\mathrm{E}(J_\theta^{(1)}(XA)) = J_\theta^{(1)}(X)\ [p_{2|\theta|/2}]J_\theta^{(1)}$$

Corollary

When A is taken from an $n \times n$ GOE,

$$\mathrm{E}(J_\theta^{(2)}(XA)) = J_\theta^{(2)}(X)\ [p_{2|\theta|/2}]J_\theta^{(2)}$$

Example Using Zonal Polynomials

$$
\begin{aligned}
Z_{[1^4]} &= 1p_{[1^4]} - 6p_{[2,1^2]} + 3p_{[2,2]} + 8p_{[3,1]} - 6p_{[4]} \\
Z_{[2,1^2]} &= 1p_{[1^4]} - p_{[2,1^2]} - 2p_{[2,2]} - 2p_{[3,1]} + 4p_{[4]} \\
Z_{[2^2]} &= 1p_{[1^4]} + 2p_{[2,1^2]} + 7p_{[2,2]} - 8p_{[3,1]} - 2p_{[4]} \\
Z_{[3,1]} &= 1p_{[1^4]} + 5p_{[2,1^2]} - 2p_{[2,2]} + 4p_{[3,1]} - 8p_{[4]} \\
Z_{[4]} &= 1p_{[1^4]} + 12p_{[2,1^2]} + 12p_{[2,2]} + 32p_{[3,1]} + 48p_{[4]}
\end{aligned}
$$

Example

$$\mathrm{E}(Z_{[3,1]}(A)) = -2(1n^4 + 5n^3 - 2n^2 + 4n^2 - 8n)$$

Example Using Zonal Polynomials

$$
\begin{aligned}
Z_{[1^4]} &= 1p_{[1^4]} - 6p_{[2,1^2]} + 3p_{[2,2]} + 8p_{[3,1]} - 6p_{[4]}\\
Z_{[2,1^2]} &= 1p_{[1^4]} - p_{[2,1^2]} - 2p_{[2,2]} - 2p_{[3,1]} + 4p_{[4]}\\
Z_{[2^2]} &= 1p_{[1^4]} + 2p_{[2,1^2]} + 7p_{[2,2]} - 8p_{[3,1]} - 2p_{[4]}\\
Z_{[3,1]} &= 1p_{[1^4]} + 5p_{[2,1^2]} - 2p_{[2,2]} + 4p_{[3,1]} - 8p_{[4]}\\
Z_{[4]} &= 1p_{[1^4]} + 12p_{[2,1^2]} + 12p_{[2,2]} + 32p_{[3,1]} + 48p_{[4]}
\end{aligned}
$$

θ	$[1^4]$	$[2,1^2]$	$[2^2]$	$[3,1]$	$[4]$
$\langle p_\theta, p_\theta\rangle_2$	$4!\cdot 2^4 = 384$	$2!\cdot 2\cdot 2^3 = 32$	$2!\cdot 2^2\cdot 2^2 = 32$	$3\cdot 2^2 = 12$	$4\cdot 2 = 8$
$\langle Z_\theta, Z_\theta\rangle_2$	2880	720	2880	2016	40320

Example

$$\langle Z_{[4]}, Z_{[4]}\rangle = 1^2\cdot 384 + 12^2\cdot 32 + 12^2\cdot 32 + 32^2\cdot 12 + 48^2\cdot 8 = 40320$$

Example Using Zonal Polynomials

$$
\begin{aligned}
Z_{[1^4]} &= 1p_{[1^4]} - 6p_{[2,1^2]} + 3p_{[2,2]} + 8p_{[3,1]} - 6p_{[4]} \\
Z_{[2,1^2]} &= 1p_{[1^4]} - p_{[2,1^2]} - 2p_{[2,2]} - 2p_{[3,1]} + 4p_{[4]} \\
Z_{[2^2]} &= 1p_{[1^4]} + 2p_{[2,1^2]} + 7p_{[2,2]} - 8p_{[3,1]} - 2p_{[4]} \\
Z_{[3,1]} &= 1p_{[1^4]} + 5p_{[2,1^2]} - 2p_{[2,2]} + 4p_{[3,1]} - 8p_{[4]} \\
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$$

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$\langle Z_\theta, Z_\theta \rangle_2$	2880	720	2880	2016	40320

Example

$$
p_{[4]} = -6\frac{8}{2880}Z_{[1^4]} + 4\frac{8}{720}Z_{[2,^2]} - 2\frac{8}{2880}Z_{[2^2]} - 8\frac{8}{2016}Z_{[3,1]} + 48\frac{8}{40320}Z_{[4]}
$$

Outline

For general β, integrate over eigenvalues

Definition

For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, define an expectation operator $\langle \cdot \rangle$ by

$$\langle f \rangle_{1+b} := c_{1+b} \int_{\mathbb{R}^n} |V(\boldsymbol{\lambda})|^{\frac{2}{1+b}} f(\boldsymbol{\lambda}) \mathrm{e}^{-\frac{1}{2(1+b)} p_2(\boldsymbol{\lambda})} \, \mathrm{d}\boldsymbol{\lambda},$$

with c_{1+b} chosen such that $\langle 1 \rangle_{1+b} = 1$.

Theorem (Okounkov - 1997)

If n is a positive integer, $1+b$ is a positive real number, and θ is an integer partition of $2n$, then

$$\left\langle J_\theta^{(1+b)}(\boldsymbol{\lambda}) \right\rangle_{1+b} = J_\theta^{(1+b)}(I_n)[p_{[2^n]}] J_\theta^{(1+b)}.$$

Jack Polynomials

	$p_{[1^4]}$	$p_{[2,1^2]}$	$p_{[2^2]}$	$p_{[3,1]}$	$p_{[4]}$
$J_{[1^4]}^{(1+b)}$	1	-6	3	8	-6
$J_{[2,1^2]}^{(1+b)}$	1	$b-2$	$-b-1$	-2	$2b+2$
$J_{[2^2]}^{(1+b)}$	1	2	b^2+3b+3	$-4b-4$	$-b^2-b$
$J_{[3,1]}^{(1+b)}$	1	$3b+2$	$-b-1$	$2b^2+2b$	$-2b^2-4b-2$
$J_{[4]}^{(1+b)}$	1	$6b+6$	$3b^2+6b+3$	$8b^2+16b+8$	$6b^3+18b^2+18b+6$

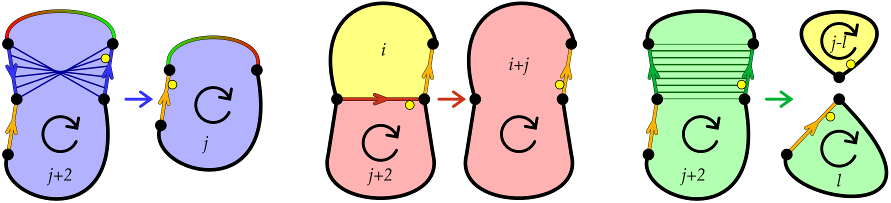
θ	$\langle J_\theta, J_\theta \rangle_{1+b}$
$[1^4]$	$24b^4+240b^3+840b^2+1200b+576$
$[2,1^2]$	$4b^5+40b^4+148b^3+256b^2+208b+64$
$[2^2]$	$8b^6+84b^5+356b^4+780b^3+932b^2+576b+144$
$[3,1]$	$12b^6+100b^5+340b^4+604b^3+592b^2+304b+64$
$[4]$	$144b^7+1272b^6+4752b^5+9744b^4+11856b^3+8568b^2+3408b+576$

A Recurrence for edge Deletion

Adapting Aomoto's proof of the Selberg integral gives an algebraic recurrence for computing $\langle p_\theta \rangle$

$$\langle p_{j+2} p_\theta \rangle = b(j+1) \langle p_j p_\theta \rangle + \alpha \sum_{i \in \theta} i m_i(\theta) \langle p_{i+j} p_{\theta \setminus i} \rangle + \sum_{l=0}^{j} \langle p_l p_{j-l} p_\theta \rangle .$$

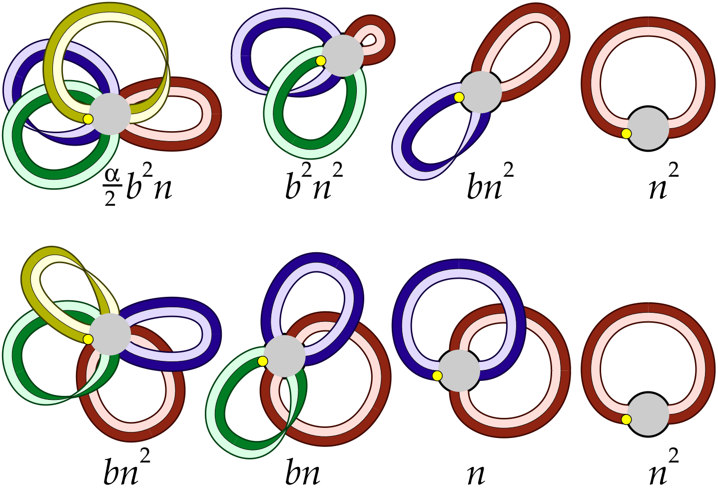
It corresponds to a combinatorial recurrence for counting polygon glueings.

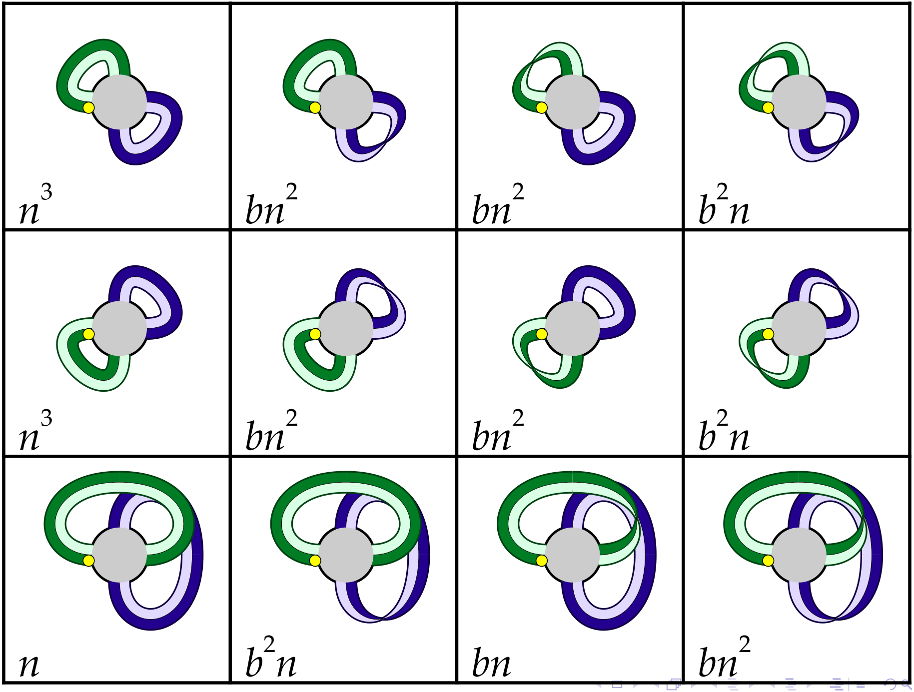


Example

Root Edge Deletion

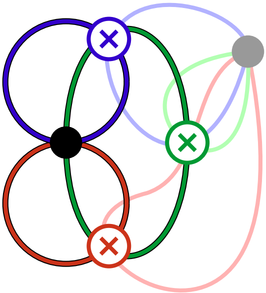
A rooted map with k edges can be thought of as a sequence of k maps.

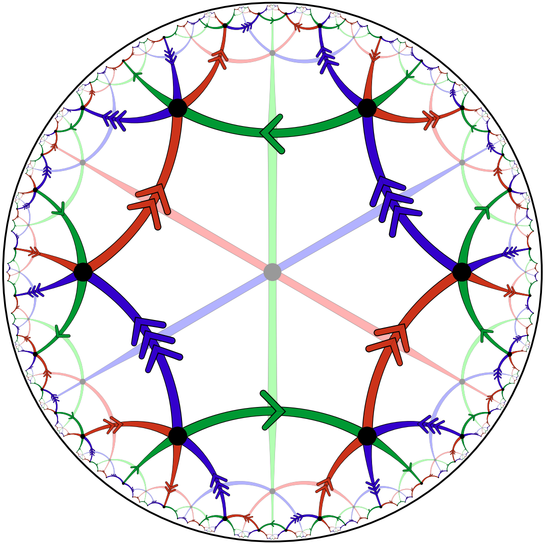



n^3
bn^2
bn^2
b^2n
n^3
bn^2
bn^2
b^2n
n
b^2n
bn
bn^2

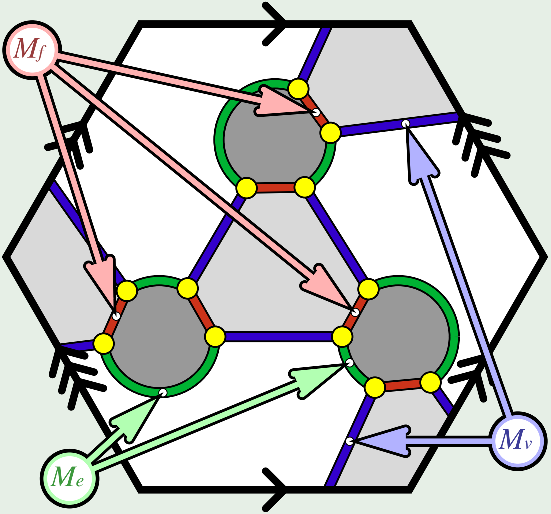
The End

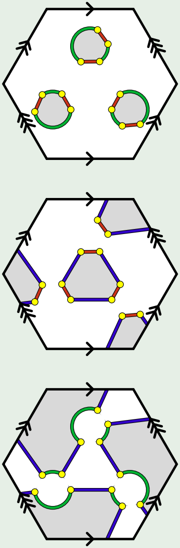
Thank You





Example





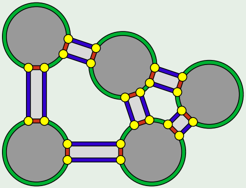
$\nu = [2^3]$

$\epsilon = [3^2]$

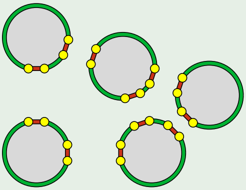
$\phi = [6]$

Return

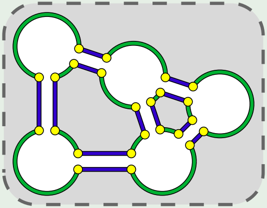
Example



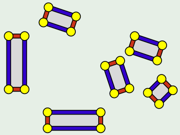
is enumerated by $\left(x_2^3\, x_3^2\right)\left(y_3\, y_4\, y_5\right)\left(z_2^6\right)$.



$\nu = [2^3, 3^2]$



$\phi = [3, 4, 5]$



$\epsilon = [2^6]$

Return

$$\langle p_{j+2}p_\theta\rangle = b(j+1)\langle p_j p_\theta\rangle + (1+b)\sum_{i\in\theta} i m_i(\theta)\langle p_{i+j}p_{\theta\setminus i}\rangle + \sum_{l=0}^{j}\langle p_l p_{j-l} p_\theta\rangle$$

Example

$$\langle 1\rangle = 1$$
$$\langle p_0\rangle = n$$
$$\langle p_2\rangle = b\langle p_0\rangle + \langle p_0 p_0\rangle = bn + n^2$$
$$\langle p_1 p_1\rangle = (1+b)\langle p_0\rangle = (1+b)n$$
$$\langle p_4\rangle = 3b\langle p_2\rangle + \langle p_0 p_2\rangle + \langle p_1 p_1\rangle + \langle p_2 p_0\rangle = (1+b+3b^2)n + 5bn^2 + 2n^3$$
$$\langle p_3 p_1\rangle = 2b\langle p_1 p_1\rangle + (1+b)\langle p_2\rangle + \langle p_0 p_1 p_1\rangle + \langle p_1 p_0 p_1\rangle = (3b+3b^2)n + (3+3b)n^2$$
$$\langle p_2 p_2\rangle = b\langle p_0 p_2\rangle + 2(1+b)\langle p_2\rangle + \langle p_0 p_0 p_2\rangle = 2b(1+b)n + (2+2b+b^2)n^2 + 2bn^3 + n^4$$
$$\langle p_2 p_{1,1}\rangle = b\langle p_0 p_{1,1}\rangle + 2(1+b)\langle p_{1,1}\rangle + \langle p_0 p_0 p_{1,1}\rangle = 2(1+b)^2 n + (b+b^2)n^2 + (1+b)n^3$$
$$\langle p_1 p_3\rangle = 3(1+b)\langle p_2\rangle = (3b+3b^2)n + (3+3b)n^2$$
$$\langle p_1 p_{2,1}\rangle = 2(1+b)\langle p_{1,1}\rangle + (1+b)\langle p_0 p_2\rangle = (2+4b+2b^2)n + (b+b^2)n^2 + (1+b)n^3$$
$$\langle p_{1,1,1,1}\rangle = 3(1+b)\langle p_0 p_{1,1}\rangle = (1+2b+b^2)n^2$$

Return